

NON-COMMUTATIVE GEOMETRY FROM STRINGS ¹

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Synopsis

One of the most important recent developments in string theory is the discovery of D-branes and a deeper understanding of their properties. Among other things, this allows one to derive for the first time noncommutative geometry from string theory. The resulting noncommutative geometry has stimulated a wide range of physical applications and has led to new insights into a fundamental formulation of string theory.

1 Noncommutative Geometry from String Theory

The first use of noncommutative geometry in string theory appears in the work of Witten on open string field theory [1] where the noncommutativity is associated with the product of open string fields. Noncommutative geometry appears in the recent development of string theory in the seminal work of Connes, Douglas and Schwarz [2] where they constructed and identified the compactification of Matrix theory on noncommutative torus.

We refer the readers to the textbooks [3–6] for general references on noncommutative geometry, [7–9] for an introduction to string theory. And to the excellent reviews [10–14] for an in-depth discussion of the physical aspects of the modern development of noncommutative geometry in string theory, as well as for a more comprehensive list of references.

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1.1 Matrix theory compactification and noncommutative geometry

M-theory is an eleven dimensional quantum theory of gravity which is believed to underlie all superstring theories. Banks, Fischler, Shenker and Susskind proposed that the large N limit of the supersymmetric matrix quantum mechanics of N D0-branes should describe the M theory compactified on a light-like circle [15]. See also [16] for review. Compactification of Matrix theory on torus can be easily achieved by considering the torus as the quotient space $\mathbb{R}^d/\mathbb{Z}^d$ with the quotient conditions

$$U_i^{-1} X^j U_i = X^j + \delta_i^j 2\pi R_i, \quad i = 1, \dots, d. \quad (1)$$

Here R_i are the radii of the torus. The unitary translation generators U_i generate the torus. They satisfy $U_i U_j = U_j U_i$. T-dualizing the D0-brane system, the equation (1) leads to the dual description as a $(d+1)$ -dimensional supersymmetric gauge theory on the dual toroidal D-brane [17]. A noncommutative torus T_θ^d is defined by the modified relations

$$U_i U_j = e^{i\theta_{ij}} U_j U_i, \quad (2)$$

where θ_{ij} specify the noncommutativity. Compactification on noncommutative torus can be easily accommodated and leads to noncommutative gauge theory [18] on the dual D-brane. The parameters θ_{ij} can be identified with the components C_{-ij} of the 3-form potential in M-theory.

Since M-theory compactified on a circle leads to IIA string theory, the components C_{-ij} corresponds to the Neveu-Schwarz (NS) B-field B_{ij} in IIA string theory. The physics of the D0-brane system in the presence of a NS B-field [19] can also be studied from the view of IIA string theory. This lead Douglas and Hull to obtain the same result that a noncommutative field theory lives on the D-brane [20]. Toroidally compactified IIA string theory has a T-duality group $SO(d, d; \mathbb{Z})$. The T-duality symmetry got translated into an equivalence relation between gauge theories on noncommutative torus [2, 21, 22]: A gauge theory on noncommutative torus T_θ^d is equivalent to a gauge theory on noncommutative torus $T_{\theta'}^d$ if their noncommutative parameters and metrics are related by the T-duality transformation. For example,

$$\theta' = (A\theta + B)(C\theta + D)^{-1}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(d, d; \mathbb{Z}). \quad (3)$$

It is remarkable that the T-duality acts within the field theory level, rather than mixing up the field theory modes with the string winding states and other stringy excitations. Mathematically (3) is precisely the condition for the noncommutative tori T_θ^d and $T_{\theta'}^d$ to be Morita equivalent [3].

1.2 Open string in B -field

It was soon realized that the D-brane is not necessarily to be toroidal in order to be non-commutative [23]. A direct canonical quantization of the open string system shows that a constant B -field on a D-brane leads to noncommutative geometry on the D-brane world-volume. Consider an open string moving in a flat space with metric g_{ij} and a constant NS B -field. In the presence of a D p -brane, the components of the B -field not along the brane can be gauged away and thus the B -field can have effects only in the longitudinal directions along the brane. The worldsheet (bosonic) action for this part is

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma (g_{ij} \partial_a x^i \partial^a x^j - 2\pi\alpha' B_{ij} \epsilon^{ab} \partial_a x^i \partial_b x^j), \quad (4)$$

where $i, j = 0, 1, \dots, p$ is along the brane. It is easy to see that the boundary condition $g_{ij} \partial_{\sigma} x^j + 2\pi i \alpha' B_{ij} \partial_{\tau} x^j = 0$ at $\sigma = 0, \pi$ is not compatible with the standard canonical quantization $[x^i(\tau, \sigma), x^j(\tau, \sigma')] = 0$ at the boundary. Taking the boundary condition as constraints and perform the canonical quantization, one obtains the commutation relations

$$[a_m^i, a_n^j] = m G^{ij} \delta_{m+n}, \quad [x_0^i, p_0^j] = i G^{ij}, \quad [x_0^i, x_0^j] = i \theta^{ij}. \quad (5)$$

Here the open string mode expansion is

$$x^i(\tau, \sigma) = x_0^i + 2\alpha' (p_0^i \tau - 2\pi\alpha' (g^{-1} B)_j^i p_0^j \sigma) + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (i a_n^i \cos n\sigma - 2\pi\alpha' (g^{-1} B)_j^i a_n^j \sin n\sigma).$$

G^{ij} and θ^{ij} are the symmetric and anti-symmetric parts of the matrix $(g + 2\pi\alpha' B)^{-1ij}$,

$$G^{ij} = \left(\frac{1}{g + 2\pi\alpha' B} g \frac{1}{g - 2\pi\alpha' B} \right)^{ij}, \quad \theta^{ij} = -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B} \right)^{ij}. \quad (6)$$

It follows from (5) that the boundary coordinates $x^i \equiv x^i(\tau, 0)$ obey the commutation relation

$$[x^i, x^j] = i \theta^{ij}. \quad (7)$$

The relation (7) implies that the D-brane worldvolume, where the opens string endpoints live, is a noncommutative manifold. One may also starts with the closed string Green function and let its arguments to approach the boundary to obtain the open string Green function

$$\langle x^i(\tau) x^j(\tau') \rangle = -\alpha' G^{ij} \ln(\tau - \tau')^2 + \frac{i}{2} \theta^{ij} \epsilon(\tau - \tau'), \quad (8)$$

where $\epsilon(\tau)$ is the sign of τ . From (8), Seiberg and Witten [24] extract the commutator (7) again. $G_{ij} = g_{ij} - (2\pi\alpha')^2 (B g^{-1} B)_{ij}$ is called the open string metric since it controls the short distance behaviour of open strings. In contrast, the short distance behaviour for

closed strings is controlled by the closed string metric g_{ij} . One may also treat the boundary B -term in (4) as a perturbation to the open string conformal field theory and from which one may extract (8) from the modified operator product expansion of the open string vertex operators [25].

D-branes in Wess-Zumino-Witten model provides another example of noncommutative geometry. In this case, the background is not flat since there is a nonzero $H = dB \sim k^{-1/2}$, where k is the level. Examining the vertex operator algebra, [26] obtains that D-branes are described by nonassociative deformation of fuzzy spheres with nonassociativity controlled by $1/k$.

1.3 String amplitudes and effective action

The effect of the B -field on the open string amplitudes is simple to determine since only the x_0^i commutation relation is affected nontrivially. For example the *noncommutative gauge theory* can be obtained from the tree level string amplitudes readily [24, 27]. For tree and one loop [28], we can use the vertex operator formalism. Generally the vertex operator can be inserted at either the $\sigma = 0$ or $\sigma = \pi$ border, where the string has the zero mode parts x_0^i and $y_0^i \equiv x_0^i - (2\pi\alpha')^2(g^{-1}B)_j^i p_0^j$ respectively. The commutation relations are

$$[x_0^i, x_0^j] = i\theta^{ij}, \quad [x_0^i, y_0^j] = 0, \quad [y_0^i, y_0^j] = -i\theta^{ij}. \quad (9)$$

The difference in the commutation relation for x_0 and y_0 implies that the two borders of the open string has opposite commutativity. This fact is not so important for tree level calculations since one can always choose to put all the interactions at, for example, the $\sigma = 0$ border. Collect all these zero mode parts of the vertex operators, one obtain a phase factor

$$e^{ip^1 x_0} e^{ip^2 x_0} \cdots e^{ip^N x_0} = e^{i \sum p^a x_0} e^{-\frac{i}{2} \sum_{I < J}^N p^I \theta p^J}, \quad (10)$$

where the external momenta p^a are ordered cyclically on the circle and momentum conservation has been used. The computation of the oscillator part of the amplitude is the same as in the $B = 0$ case, except that the metric G is employed in the contractions. As a result, the effect of B -field on the tree level string amplitude is simply to multiply the amplitude at $B = 0$ with the phase factor and to replace the metric by the metric G . A generic term in the tree level effective action simply becomes

$$\int d^{p+1}x \sqrt{-\det g} \operatorname{tr} \partial^{n_1} \Phi_1 \cdots \partial^{n_k} \Phi_k \rightarrow \int d^{p+1}x \sqrt{-\det G} \operatorname{tr} \partial^{n_1} \Phi_1 * \cdots * \partial^{n_k} \Phi_k. \quad (11)$$

Here the star-product, also called the Moyal product is defined by

$$(f * g)(x) = e^{i \frac{\theta^{\mu\nu}}{2} \frac{\partial}{\partial x_1^\mu} \frac{\partial}{\partial x_2^\nu}} f(x_1) g(x_2) |_{x_1=x_2}. \quad (12)$$

The star-product is associative, noncommutative and satisfies $\overline{f * g} = \bar{g} * \bar{f}$ under complex conjugation. Also for functions vanish rapidly enough at infinity, it is

$$\int f * g = \int g * f = \int fg. \quad (13)$$

An interesting consequence of the nonlocality as expressed by the noncommutative geometry (7) is the existence of dipole excitation [29] whose extent is proportional to its momentum, $\Delta x = k\theta$. This relation is at the heart of the *IR/UV mixing phenomena* (see below) of noncommutative field theory. Moyal product in string theory has also appeared earlier in the works [30].

At one (and higher) loop level, the different noncommutativity for the opposite boundaries of open string becomes essential and give rises to new effects [31]. In this case non-planar diagrams require to put vertex operators at the two different borders $\sigma = 0, \pi$. A more complicated phase factor which involves internal as well as external momentum is resulted. This leads to the IR/UV mixing in the noncommutative quantum field theory. The different noncommutativity for the opposite boundaries of open string (9) is the basic reason for the IR/UV mixing in the noncommutative quantum field theory. The commutation relations (5) are valid at all loops and therefore one can use them to construct the higher loop string amplitudes from the first principle [31]. The effect of the B -field on the string interaction can be easily implemented into the Reggeon vertex and the complete higher loop amplitudes in B-field have been constructed.

1.4 Low energy limit: the Seiberg-Witten limit and the NCOS limit

The full open string system is still quite complicated. One may try to decouple the infinite number of massive string modes to obtain a low energy field theoretic description by taking the limit $\alpha' \rightarrow 0$. Since open string are sensitive to G and θ , one should take the limit such that G and θ are fixed. For the magnetic case $B_{0i} = 0$, Seiberg and Witten [24] showed that this can be achieved with the following double scaling limit

$$\alpha' \sim \epsilon^{1/2}, \quad g_{ij} \sim \epsilon \rightarrow 0 \quad (14)$$

with B_{ij} and everything else kept fixed. Assuming B is of rank r , then (6) becomes

$$G_{ij} = -(2\pi\alpha')^2 (Bg^{-1}B)_{ij}, \quad \theta^{ij} = (B^{-1})^{ij}, \quad \text{for } i, j = 1, \dots, r. \quad (15)$$

Otherwise $G_{ij} = g_{ij}$, $\theta^{ij} = 0$. One may also argue that closed string decouples in this limit. As a result, we obtain at the low energy limit a greatly simplified noncommutative Yang-Mills action $F * F$. See below for more discussion of this field theory.

For the case of a constant electric field background, say $B_{01} \neq 0$, there is a critical electric field beyond which open string become unstable and the theory does not make sense. Due to the presence of this upper bound of the electric field, one can show that there is no decoupling limit where one can reduce the string theory to a field theory on a noncommutative spacetime. However one can consider a different scaling limit where one takes the closed string metric is scale to infinity appropriately as the electric field approaches the critical value. In this limit, all closed string modes decoupled. One obtains a novel noncritical string theory living on a noncommutative spacetime known as Noncommutative Open String (NCOS). For more details, see [32, 33]

2 Noncommutative Quantum Field Theory

Field theory on noncommutative spacetime are defined using the star-product instead of ordinary product of the fields. To illustrate the general ideas, let us consider a single real scalar field theory with the action

$$S = \int d^D x \left[\frac{1}{2} (\partial\phi)^2 - \frac{m^2}{2} \phi^2 - V(\phi) \right], \quad V(\phi) = \frac{g}{4!} \phi^{*4}. \quad (16)$$

Due to the property (13), free noncommutative field theory is the same as an ordinary field theory. Treating the interaction term as perturbation, one can perform the usual quantization and obtain the Feynman rules: the propagator is unchanged and the interaction vertex in the momentum space is given by g times the phase factor

$$\exp \left(-\frac{i}{2} \sum_{1 \leq a < b \leq 4} p^a \times p^b \right). \quad (17)$$

Here $p \times q \equiv p_\mu \theta^{\mu\nu} q_\nu$. The theory is nonlocal due to the infinite order of derivatives that appear in the interaction.

2.1 Planar and non-planar diagrams

The factor (17) is cyclically symmetric but not permutation symmetric. This is analogous to the situation of a matrix field theory. Using the same double line notation as introduced by 't Hooft, one can similarly classify the Feynman diagrams of noncommutative field theory according to its genus. In particular the total phase factor of a planar diagram behaves quite

differently from that of a non-planar diagram. It is easy to show that a planar diagram will have the phase factor

$$V_p(p^1, \dots, p^n) = \exp \left(-\frac{i}{2} \sum_{1 \leq a < b \leq n} p^a \times p^b \right), \quad (18)$$

where p^1, \dots, p^n are the (cyclically ordered) external momenta of the graph. Note that the phase factor (18) is independent of the internal momenta. This is not the case for a non-planar diagram. One can easily show that a non-planar diagram carries an additional phase factor [34]

$$V_{np} = V_p \exp \left(-\frac{i}{2} \sum_{1 \leq a < b \leq n} C_{ab} p^a \times p^b \right), \quad (19)$$

where the C_{ab} is the signed intersection matrix of the graph, whose ab matrix element counts the number of times the a -th (internal or external) line crosses the b -th line. The matrix C_{ab} is not uniquely determined by the diagram as different ways of drawing the graph could lead to different intersections. However the phase factor (19) is unique due to momentum conservation.

This different behaviour of the planar and non-planar phase factor has important consequences.

1. Since the phase factor (18) is independent of the internal momenta, the divergences and renormalizability of the planar diagrams will be (simply) the same as in the commutative theory and can be handled with standard renormalization techniques. This is sharply different for the non-planar diagrams. In fact due to the extra oscillatory internal momenta dependent phase factor, one can expect the non-planar diagrams to have an improved ultra-violet (UV) behaviour. It turns out that planar and non-planar diagrams also differ sharply in their infrared (IR) behaviour due to the IR/UV mixing effect.

2. Moreover at high energies, one can expect noncommutative field theory will generically become planar since the non-planar diagrams will be suppressed due to the oscillatory phase factor.

3. In the limit $\theta \rightarrow \infty$, the non-planar sector will be totally suppressed since the rapidly oscillating phase factor will cause the non-planar diagram to vanish upon integrating out the momenta. Thus generically the large θ limit is analogous to the large N limit where only the planar diagrams contribute. However these expectations do not apply for noncommutative gauge theory since one need to include *open Wilson line* (see below) in the construction of gauge invariant observables, and the open Wilson line grows in extent with energy and θ .

2.2 IR/UV mixing

Due to the nonlocal nature of noncommutative field theory, there is generally a mixing of the UV and IR scales [35]. The reason is roughly the follow. Non-planar diagrams generally has phase factors like $\exp(ik\theta p)$, with k a loop momentum, p an external momentum. Consider a non-planar diagram which is UV divergent when $\theta = 0$, one can expect that for very high loop momenta, the phase factor will oscillate rapidly and renders the integral finite. However this is only valid for a non-vanishing external momentum θp , the infinity will come back as $\theta p \rightarrow 0$. However this time it appears as an IR singularity. Thus an IR divergence arises whose origin is from the UV region of the momentum integration and this is known as the IR/UV mixing phenomena.

To be more specific, consider the ϕ^4 scalar theory in $D = 4$ dimensions. The one loop self energy has a non-planar contribution given by

$$\Gamma_{\text{np}} = \frac{g}{6(2\pi)^4} \int \frac{d^4 k}{k^2 + m^2} e^{ik\theta p} \sim \frac{g}{3(4\pi^2)^2} (\Lambda_{\text{eff}}^2 + \dots), \quad (20)$$

where $\Lambda_{\text{eff}}^2 = (1/\Lambda^2 + (\theta p)^2)^{-1}$. One can see clearly the IR/UV mixing: Γ_{np} is UV finite as long as $\theta p \neq 0$; when $\theta p = 0$, the quadratic UV divergence is recovered, $\Gamma_{\text{np}} \sim \Lambda^2$. For supersymmetric theory, we have at most logarithmic IR singularities from IR/UV mixing.

IR/UV mixing has a number of interesting consequences.

1. Due to the IR/UV mixing, noncommutative theory does not appear to have a consistent Wilsonian description since it will require that correlation functions computed at finite Λ differ from their limiting values by terms of order $1/\Lambda$ for all values of momenta. However this is not true for theory with IR/UV mixing. For example, the two point function (20) at finite value of Λ differs from its value at $\Lambda = \infty$ by the amount $\Gamma_{\text{np}}^\Lambda - \Gamma_{\text{np}}^{\Lambda=\infty} \propto 1/(\theta p)^2$, for the range of momenta $(\theta p)^2 \ll 1/\Lambda^2$. It has been argued that the IR singularity may be associated with missing light degrees of freedom in the theory. With new degrees of freedom appropriately added, one may recover a conventional Wilsonian description. Moreover it has been suggested that to identify these degrees of freedom with the closed string modes [35]. However the precise nature and origin of these degrees of freedom is not known.

2. The renormalization of the planar diagrams is straight forward, however it is more subtle for the non-planar diagrams since the IR/UV-mixed IR singularities may mix with other divergences at higher loops and render the proof of renormalizability much more difficult. IR/UV mixing renders certain large N noncommutative field theory nonrenormalizable [36]. However for theories with a fixed set of degree of freedoms to start with, it is believed that one can have sufficiently good control of the IR divergences and prove renormalizability.

An example of renormalizable noncommutative quantum field theory is the noncommutative Wess-Zumino model [37] where IR/UV mixing is absent. However a general proof is still lacking despite the progress made in [38].

3. One can show that IR/UV mixing in timelike noncommutative theory ($\theta^{0i} \neq 0$) leads to breakdown of perturbative unitarity [39–41]. For theory without IR/UV mixing, unitarity will be respected even if the theory has a timelike noncommutativity [41]. Theory with lightlike noncommutativity is unitary [42].

3 Noncommutative Gauge Theory

Gauge theory on noncommutative space is defined by the action

$$S = -\frac{1}{4g^2} \int dx \operatorname{tr} \left(F_{ij}(x) * F^{ij}(x) \right), \quad (21)$$

where the gauge fields A_i are $N \times N$ Hermitian matrices, F_{ij} is the noncommutative field strength $F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i, A_j]_*$ and tr is the ordinary trace over $N \times N$ matrices. The theory is invariant under the star-gauge transformation

$$A_i \rightarrow g * A_i * g^\dagger - ig * \partial_i g^\dagger, \quad (22)$$

where the $N \times N$ matrix function $g(x)$ is unitary with respect to the star-product $g * g^\dagger = g^\dagger * g = I$. The solution is $g = e_*^{i\lambda}$ where λ is Hermitian. In infinitesimal form, $\delta_\lambda A_i = \partial_i \lambda + i[\lambda, A_i]_*$. The noncommutative gauge theory has N^2 Hermitian gauge fields. Because of the star-product, the $U(1)$ sector of the theory is not free and does not decouple from the $SU(N)$ factor as in the commutative case. Note that this way of defining noncommutative gauge theory does not work for the other Lie group since the star-commutator generally involves commutator as well as anti-commutator of the Lie algebra, and hence the expressions above generally involve the enveloping algebra of the underlying Lie group. With the help of the *Seiberg-Witten map* (see below), one can construct an enveloping algebra valued gauge theory which has the same number of independent gauge fields and gauge parameters as the ordinary Lie algebra valued gauge theory [43]. However the quantum properties of these theories are much less understood. One may also introduce certain automorphism in the noncommutative $U(N)$ theory to restrict the dependence of the noncommutative space coordinates of the field configurations and obtain a notion of noncommutative theory with orthogonal and symplectic star-gauge group [44]. However the theory does not reduce to the standard gauge theory in the commutative limit $\theta \rightarrow 0$.

3.1 Open Wilson line and gauge invariant observables

One remarkable feature of noncommutative gauge theory is the mixing of noncommutative gauge transformation and spacetime translation, as can be seen from the following identity

$$e^{ikx} * f(x) * e^{-ikx} = f(x + k\theta), \quad (23)$$

for any function f . This is analogous to the situation in general relativity where translations are also equivalent to gauge transformations (general coordinate transformations). Thus as in general relativity, there are no local gauge invariant observables in noncommutative gauge theory. The unification of spacetime and gauge fields in noncommutative gauge theory can also be seen from the fact that derivatives can be realized as commutator $\partial_i f \rightarrow -i[\theta_{ij}^{-1} x^j, f]$ and get absorbed into the vector potential in the covariant derivative

$$D_i = \partial_i + iA_i \rightarrow -i\theta_{ij}^{-1} x^j + iA_i. \quad (24)$$

Equation (24) clearly demonstrates the unification of spacetime and gauge fields. Note that the field strength takes the form $F_{ij} = i[D_i, D_j] + \theta_{ij}^{-1}$.

The Wilson line operator for a path C running from x_1 to x_2 is defined by

$$W(C) = P_* \exp(i \int_C A). \quad (25)$$

P_* denotes the path ordering with respect to the star-product, with $A(x_2)$ at the right. It transforms as

$$W(C) \rightarrow g(x_1) * W(C) * g(x_2)^\dagger. \quad (26)$$

In commutative gauge theory, the Wilson line operator for closed loop (or its Fourier transform) is gauge invariant. In noncommutative gauge theory, the closed Wilson loops are no longer gauge invariant. Noncommutative generalization of the gauge invariant Wilson loop operator can be constructed most readily by deforming the Fourier transform of the Wilson loop operator. It turns out that the closed loop has to open in a specific way to form an open Wilson line in order to be gauge invariant [45]. To see this, let us consider a path C connecting points x and $x + l$. Using (23), it is easy to see that the operator

$$\tilde{W}(k) \equiv \int dx \operatorname{tr} W(C) * e^{ikx}, \quad \text{with} \quad l^j = k_i \theta^{ij}, \quad (27)$$

is gauge invariant. Just like Wilson loops in ordinary gauge theory, these operators also constitute an over complete set of gauge invariant operators parametrized by the set of curve C . When $\theta = 0$, C become a closed loop and we re-obtain the (Fourier transform) usual closed Wilson loop in commutative gauge theory. Noncommutative version of the loop equation for

closed Wilson loop has been constructed and involves open Wilson line [46]. The open Wilson line is instrumental in the construction of gauge invariant observables [47–50]. An important application is in the construction of various couplings of the noncommutative D-brane to the bulk supergravity fields [51]. The equivalence of the commutative and noncommutative couplings to the RR-fields has led to the exact expression for the Seiberg-Witten map [52]. It is remarkable that the one-loop nonplanar effective action for noncommutative scalar theory [53], gauge theory [54–56] as well as the two-loop effective action for scalar [57] can be written compactly in terms of open Wilson line. Based on this result, the physical origin of the IR/UV mixing has been elucidated. One may identify the open Wilson line with the dipole excitation generically presents in noncommutative field theory [53, 57] and hence explain the presence of the IR/UV mixing. IR/UV mixing may also be identified with the instability [54, 55] associated with the closed string exchange of the noncommutative D-branes.

3.2 The Seiberg-Witten map

The open string is coupled to the one-form A_i living on the D-brane through the coupling $\int_{\partial\Sigma} A$. For slowly varying fields, the effective action for this gauge potential can be determined from the S-matrix and is given by the Dirac-Born-Infeld (DBI) action. In the presence of B -field, the discussion above (11) leads to the noncommutative DBI Lagrangian

$$L_{NCDBI}(\hat{F}) = G_s^{-1} \mu_p \sqrt{-\det(G + 2\pi\alpha' \hat{F})}, \quad (28)$$

where $\mu_p = (2\pi)^{-p}(\alpha')^{-(p+1)/2}$ is the D-brane tension and \hat{F} is the noncommutative field strength. However one may also exploit the tensor gauge invariance on the D-brane (that is, the string sigma model is invariant under $A \rightarrow A - \Lambda, B \rightarrow B + d\Lambda$) and consider the combination $F + B$ as a whole. In this case it is like having the open string coupled to the boundary gauge field strength $F + B$ and there is no B field. And we have the usual DBI Lagrangian [58]

$$L_{DBI}(F) = g_s^{-1} \mu_p \sqrt{-\det(G + 2\pi\alpha' (F + B))}. \quad (29)$$

In (28) and (29), G_s and g_s are the effective open string couplings in the noncommutative and commutative descriptions. Although looks quite different, Seiberg and Witten showed the commutative and noncommutative DBI actions are indeed equivalent if the open string couplings are related by $g_s = G_s \sqrt{\det(g + 2\pi\alpha' B) / \det G}$ and there is a field redefinition that relate the commutative and noncommutative gauge fields. The map $\hat{A} = \hat{A}(A)$ is called the Seiberg-Witten map [24]. Moreover, the noncommutative gauge symmetry is equivalent to the ordinary gauge symmetry in the sense that they have the same set of orbits under gauge transformation:

$$\hat{A}(A) + \hat{\delta}_\lambda \hat{A}(A) = \hat{A}(A + \delta_\lambda A). \quad (30)$$

Here \hat{A}_i and $\hat{\lambda}$ are the noncommutative gauge field and noncommutative gauge transformation parameter, A_i and λ are the ordinary gauge field and ordinary transformation parameter. The map between \hat{A}_i and A_i is called the Seiberg-Witten map. (30) can be solved only if the transformation parameter $\hat{\lambda} = \hat{\lambda}(\lambda, A)$ is field dependent. The Seiberg-Witten map is characterized by the Seiberg-Witten differential equation,

$$\delta \hat{A}_i(\theta) = -\frac{1}{4} \delta \theta^{kl} [\hat{A}_k * (\partial_l \hat{A}_i + \hat{F}_{li}) + (\partial_l \hat{A}_i + \hat{F}_{li}) * \hat{A}_k]. \quad (31)$$

An exact solution for the Seiberg-Witten map can be written down with the help of the open Wilson line. For the case of $U(1)$ with constant F , we have the exact solution $\hat{F} = (1 + F\theta)^{-1} F$.

That there is a field redefinition that allows one to write the effective action in terms of different fields with different gauge symmetries may seem puzzling in the first instant. However it has a clear physical origin in terms of the string worldsheet. In fact there are different possible schemes to regularize the short distance divergence on the worldsheet. One can show that the Pauli-Villars regularization gives the commutative description, while the point splitting regularization gives the noncommutative description. Since theories defined by different regularization schemes are related by a coupling constant redefinition, this implies the commutative and noncommutative description are related by a field redefinition because the couplings on the worldsheet are just the spacetime fields.

Despite this formal equivalence, the physics of the noncommutative theories is generally quite different from the commutative one. First it is clear that generally the Seiberg-Witten map may take nonsingular configurations to singular configurations. Second, the observables one is interested in are also generally different. Moreover the two descriptions are generally good for different regimes: the conventional gauge theory description is simpler for small B and noncommutative description is simpler for large B .

3.3 Perturbative gauge theory dynamics

The noncommutative gauge symmetry (22) can be fixed as usual by employing the Fadeev-Popov procedure, resulting in Feynman rules that are similar to the conventional gauge theory. The important difference is that now we have to amend to the structure constant the phase factors (18) and (19). It turns out that the non-planar $U(N)$ diagrams contribute (only) to the $U(1)$ part of the theory. As a result, unlike the commutative case, the $U(1)$ part of the theory is no longer decoupled and free [59]. Noncommutative gauge theory is one-loop renormalizable [60, 61]. The beta-function is determined solely by the planar diagrams and,

at one loop, is given by [60–62]

$$\beta(g) = -\frac{22}{3} \frac{Ng^3}{16\pi^2}, \quad \text{for } N \geq 1. \quad (32)$$

Note that the beta function is independent of θ , the noncommutative $U(1)$ is asymptotically free and does not reduce to the commutative theory when $\theta \rightarrow 0$. Noncommutative theory beyond the tree level is generally not smooth in the limit $\theta \rightarrow 0$. Discontinuity of this kind was also noted for the Chern-Simon system [63].

Gauge anomalies can be similarly discussed [64–66] and satisfy the noncommutative generalizations of the Wess-Zumino consistency conditions [66]. In $d = 2n$ dimensions, the anomaly involves the combination $\text{tr}(T^{a_1} T^{a_2} \dots T^{a_{n+1}})$ rather than the usual symmetrized trace since the phase factor is not permutation symmetric. As a result, the usual cancellation of anomaly does not work and is the main obstacle to the construction of noncommutative chiral gauge theory.

There is a number of interesting features to mention for the IR/UV mixing [62, 67] in noncommutative gauge theory: 1. IR/UV mixing generically yields pole like IR singularities. Despite the appearance of IR poles, gauge invariance of the theory is not endangered [67]. 2. One can show that only the $U(1)$ sector is affected by IR/UV mixing. 3. As a result of IR/UV mixing, noncommutative $U(1)$ photons polarized in the noncommutative plane will have different dispersion relations from those which are not [67]. Strange as it is, this is consistent with gauge invariance.

4 Noncommutative Solitons, Instantons and D-branes

Solitons and instantons play important roles in the nonperturbative aspects of field theory. The nonlocality of star-product gives noncommutative field theory a stringy nature. It is remarkable that this applies to the nonperturbative sector as well. Solitons and instantons in the noncommutative gauge theory reproduce amazingly the properties of D-brane in the string. The reviews [11, 12] are recommended.

4.1 GMS solitons

Derrick theorem says that commutative scalar field theories in two or higher dimensions do not admit any finite energy classical solution. This follows from a simple scaling argument,

which will fail when the theory becomes noncommutative since noncommutativity introduce a fixed length scale $\sqrt{\theta}$. Noncommutative solitons in pure scalar theory can be easily constructed at the limit $\theta = \infty$. For example, consider a (2+1)-dimensional single scalar theory with a potential V and noncommutativity $\theta^{12} = \theta$. In the limit $\theta = \infty$, the potential term dominates and the noncommutative solitons are determined by the equation

$$\partial V / \partial \phi = 0. \quad (33)$$

Equation (33) can be easily solved in terms of projector. Assuming V has no linear term, the general soliton (up to unitary equivalence) is

$$\phi = \sum_i \lambda_i P_i, \quad (34)$$

where λ_i are the roots of $V'(\lambda) = 0$ and P_i is a set of orthogonal projectors. For real scalar field theory, we have to restrict the sum to real roots only. These solutions are known as the Gopakumar-Minwalla-Strominger (GMS) solitons [68]. A simple example of projector is given by $P = |0\rangle\langle 0|$, which corresponds to a Gaussian profile in the x^1, x^2 plane with width $\sqrt{\theta}$. The soliton continues to exist until θ decreases below a certain critical θ_c .

New solution can be generated from known one using the so called solution generating technique. If ϕ is a solution of (33), then

$$\phi' = T^\dagger \phi T \quad (35)$$

is also a solution provided that $TT^\dagger = 1$. In an infinite dimensional Hilbert space, T is not necessary to be unitary, i.e. $T^\dagger T \neq 1$. In this case, T is said to be a partial isometry. The new solution ϕ' is different from ϕ since they are not related by global transformation of basis.

4.2 Tachyon condensation and D-branes

A beautiful application of the noncommutative soliton is in the construction of D-brane as solitons of the tachyon field in noncommutative open string theory [69]. For the bosonic string theory, one may consider it a space filling D25-brane. Integrating out the massive string modes lead to an effective action for the tachyon and the massless gauge field A_μ . We remark that different from the pure scalar case, noncommutative soliton can be constructed exactly for finite θ in a system with gauge and scalar fields [70]. Although the detailed form of the effective action is unknown, one has enough confidence to say what the true vacuum configuration is according to the Sen conjecture. See for example, [71] for excellent review on this subject. One can then apply the solution generating technique to generate

new soliton solution. With a B -field of rank $2k$, one can construct this way solutions which are localized in \mathbb{R}^{2k} and represents a $D(25-2k)$ -brane. This is supported by the matching of the tension and the spectrum of fluctuation around the soliton configuration. Similar ideas can also be applied to construct D-branes in type II string theory. Again the starting point is an unstable brane configuration with tachyon field(s). There are two types of unstable D-branes: non-BPS Dp -branes (p odd in IIA theory and p even in IIB theory) and BPS branes anti-branes $Dp\text{-}\overline{Dp}$ system. A similar analysis allows one to identify the noncommutative soliton with the lower dimensional BPS D-branes which arises from tachyon condensation.

One main motivation for studying tachyon condensation in open string theory is the hope that open string theory may provide a fundamental nonperturbative formulation of string theory. It may not be too surprising that D-branes can be obtained in terms of open string fields. However to describe closed string and NS-branes in terms of open string degree of freedom remain an obstacle.

4.3 Noncommutative instanton and monopoles

Instantons on noncommutative \mathbf{R}_θ^4 can be constructed readily using the ADHM formalism by modifying the ADHM constraints with a constant additive term [72]. The result is that the self-dual (resp. anti-self-dual) instanton moduli space depends only on the anti-self-dual (resp. self-dual) part. The construction go through even in the $U(1)$ case. Consider a self-dual θ , the ADHM constraints for the self-dual instanton is the same as in the commutative case, and there is no nonsingular solution. On the other hand, the ADHM constraints for the anti-self-dual instanton get modified and admit nontrivial solution. This noncommutative instanton solution is nonsingular with size $\sqrt{\theta}$. Although ADHM method does not give a self-dual instanton, a direct construction can be applied to obtain non ADHM self-dual instantons. Recall that the gauge field strength can be written as $F_{ij} = i[D_i, D_j] + \theta_{ij}^{-1}$, where D_i is given by the function on the right hand side of (24). Thus a simple self-dual solution can be constructed with

$$D_i = i\theta_{ij}^{-1}T^\dagger x^j T, \quad (36)$$

where T is a partial isometry which satisfies $TT^\dagger = 1$, but $T^\dagger T = 1 - P$ is not necessarily the identity. It is clear that P is a projector. The field strength

$$F_{ij} = \theta_{ij}^{-1}P \quad (37)$$

is self-dual and has an instanton number n where n is the rank of the projector. The noncommutative instanton represent a $D(p-4)$ -brane within a Dp -brane. The ADHM constraints are just the D-flatness condition for the D-brane worldvolume gauge theory. The additive constant to the ADHM constraints also has a simple interpretation as a Fayet-Illiopolous parameter which appears in the presence of B -field.

On noncommutative \mathbb{R}^3 (say $\theta^{12} = \theta$), BPS monopoles satisfy the Bogomolny equation [73]

$$\nabla_i \Phi = \pm B_i, \quad i = 1, 2, 3 \quad (38)$$

and can be obtained by solving the Nahm equation

$$\partial_z T_i = \epsilon_{ijk} T_j T_k + \delta_{i3} \theta. \quad (39)$$

T_i are $k \times k$ Hermitian matrices depending on an auxiliary variable z and k gives the charge of the monopole. Noncommutativity modifies the Nahm equation with a constant term, which can be absorbed by a constant shift of the generators. Therefore unlike the case of instanton, the monopole moduli space is not modified by noncommutative deformation. The Nahm construction has clear physical meaning in string theory. The monopole (electric charge) can be interpreted as a D-string (fundamental string) ending on a D3-brane. One can also suspend k D-string between a collection of N parallel D3-branes, this would correspond to a charge k monopole in a Higgsed $U(N)$ gauge theory. The matrices X^i correspond to the matrix transverse coordinates of the D-strings which lie within the D3-branes.

5 Further Topics

Finally we include a brief discussions of some further topics of interests.

1. The noncommutative geometry we discuss here is of canonical type. Other deformations exists, for example, kappa-deformation [74] and fuzzy sphere which are of the Lie-algebra type, quantum group deformation [75] which is a quadratic type deformation: $x^i x^j = q^{-1} \hat{R}_{kl}^{ij} x^k x^l$, whose consistency is guaranteed by the Yang-Baxter equation. It is interesting to see whether these noncommutative geometries arise from string theory. Another natural generalization is to consider noncommutative geometry of superspace. A simple example is to consider the fermionic coordinates to be deformed with the nonvanishing relation

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}, \quad (40)$$

where $C^{\alpha\beta}$ are constants. It has been shown that (40) arises in certain Calabi-Yau compactification of type IIB string theory in the presence of RR-background [76]. The deformation (40) reduces the number of supersymmetries to half. Therefore it is called $\mathcal{N} = 1/2$ supersymmetry [77]. The noncommutativity (40) can be implemented on the superspace $(y^i, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ as a star-product for the θ^α 's. Unlike the bosonic deformation which involves an infinite number of higher derivatives, the star-product for (40) stops at order C^2 due to the Grassmanian nature of the fermionic coordinates. Field theory with $\mathcal{N} = 1/2$ supersymmetry is local and differs from the ordinary $\mathcal{N} = 1$ theory by only a few number of supersymmetry breaking

terms. The $N = 1/2$ WZ model is renormalizable [78] if extra F and F^3 terms are added to the original Lagrangian, where F is the auxiliary field. The $N = 1/2$ gauge theory is also renormalizable [79].

2. Integrability of a theory provides valuable information beyond the perturbative level. An integrable field theory is characterized by an infinite number of conserved charges in involution. It is natural to ask whether integrability is preserved by noncommutative deformation. Integrable noncommutative integrable field theories have been constructed, see [80]. In the commutative case, Ward [81] has conjectured that all (1+1) and (2+1)-dimensional integrable system can be obtained from the 4-dimensional self-dual Yang-Mills equation by reduction. Validity of the noncommutative version of the Ward conjecture has been confirmed so far [80], [82]. It is interesting to see whether it is true in general.

3. Locality and Lorentz symmetry form the corner stones of quantum field theory and standard model physics of particles. Noncommutative field theory provides a theoretical framework where one can discuss about effects of nonlocality and Lorentz symmetry violation. Possible phenomenological signals [83] have been investigated (mostly at the tree level) and bound has been placed on the extent of noncommutativity. Proper understanding and a better control of the IR/UV mixing remains the crux of the problem. Noncommutative geometry may also be relevant for cosmology and inflation [84].

4. Like the standard AdS/CFT correspondence [85], the noncommutative gauge theory should also has a gravity dual description [86, 87]. The supergravity background can be determined by considering the decoupling limit of D-branes with a NS B-field background. However since the noncommutative gauge theory does not permit any conventional local gauge invariant observable, the usual AdS/CFT correspondence that relates field theory correlators with bulk interaction does not seem to apply. It has been argued that [49] generic properties like the dipole relation between length and momentum for open Wilson line can be seen from the gravity side. A more precise understanding of the duality map is called for.

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See also: Superstring theories. Brane construction of gauge theories. Dualities in string theories. String field theory. String as large N-limit of gauge theory. Noncommutative tori, Yang-Mills and string theory. Deformation quantization. Soliton and other extended field configurations.